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48. 254.

GEOMETRICAL PROBLEMS

IN THE PROPERTIES OF THE CONIC SECTIONS.

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PREFACE.

SINCE, by the recent alterations in the Senate-House Examination, Candidates for Honors are required to be able to prove the properties of the Conic Sections by Geometrical methods, it appeared to the Author that a Collection of Problems soluble by such means would be found useful.

Every question in the following pages has been worked out geometrically, but some theorems have been introduced, in which the result is put under an ungeometrical form, either as being more familiar or more symmetrical (*e. g.* Prob. 66, *Ellipse*, and Prob. 94, *Parabola*): in these it is the last step only which is not geometrical.

Answers have been always given, and Solutions in cases where the property was so important as to be necessary for a thorough knowledge of the subject, or where any useful method of proceeding could be exemplified.

Some propositions have also been proved as being of extensive application in the solution of Problems in Central Forces, without the Differential Calculus, because it was the Author's intention to accompany these pages by a series of illustrations of the three first Sections of Newton, which he has since resolved on publishing separately.

Many of the problems have been taken from the works of Emerson, Wallace, and other geometrical writers on the subject, as well as from various Cambridge and Dublin Examination Papers. Such as were suitable in the Senate-House Papers of 1848 were inserted, with their solutions, while the work was in the press, excepting one in the paper of problems on early subjects, which is included in Prob. 31, *Ellipse:* with the exception of these, and a few others added at the same time, the questions have been as much as possible arranged progressively.

Trinity Hall, February 1848.

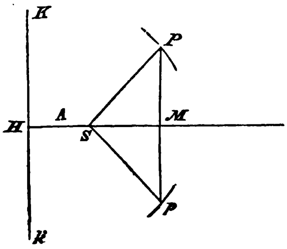
CONIC SECTIONS.

THE PARABOLA.

WHEREVER, in what follows, a parabola is required to be described, fulfilling certain conditions, the problem may be considered to be solved when the focus and directrix are found, as then any number of points may be found in the curve, as will be shewn.

1. Given the focus and directrix; to determine any number of points in the curve.

Let S be the focus and KHR the directrix. From S draw SH perpendicular to KR, and produce it indefinitely in direction HS. Bisect SH in A, take any point M in this line, not between A and H; through M draw an indefinite straight line PMp perpendicular to HM, with centre S and radius HM, describe a circle cutting PMp in P and p; P and p will be points in the parabola. For

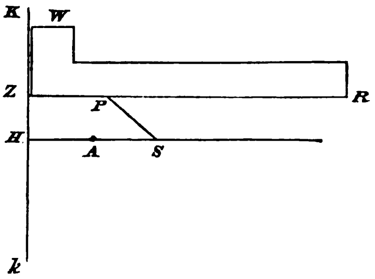


$$SP = Sp = HM = HA + AM = AS + AM,$$

the property of a parabola where the focus is S and vertex A.

2. Given the focus and directrix; shew how the parabola may be described practically.

Let WZR be a ruler, having a right angle at Z, and let the end ZW slide along the line KHk. Let one end of a string, whose length is RZ, be fixed at R and the other k at the focus S. Then if P be a moveable pencil, always so applied as to keep the string stretched, it will trace out a parabola.

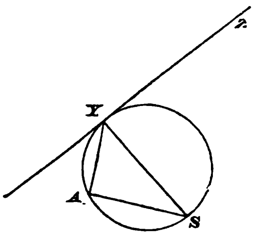


For $SP + PR$ = length of string = $ZR = ZP + PR$, therefore $SP = ZP$; the property of a parabola whose focus is S and whose directrix is KHk.

The above is a simple case of finding a geometrical *locus*. In finding the locus of a point moving according to a fixed law, we must endeavour to find some distinguishing property of a known curve (in what follows a circle or a Conic Section) which is possessed by the point in all its positions. This curve is said to be the *locus* of the point.

3. Find the locus of the vertices of all the parabolas which have a given focus and touch a given straight line.

Let S be the focus, YZ the straight line given in position. Draw SY perpendicular upon YZ; then the tangent at the vertex A of any parabola, to which YZ is a tangent, must pass through Y; also SA must be perpendicular to it. Hence that A may be the vertex of one of the required parabolas, it must be the vertex of a right-angled triangle described upon SY; all which vertices lie in the circle described on SY. This circle is therefore the locus of the vertices of the parabolas described as required.



Sometimes when the position of a point is required, which has to fulfil two conditions, the locus corresponding to each condition may be constructed, and the point or points are determined by the intersection of the loci. This method and the result of problem (3) are useful in what follows.

4. Describe a parabola with a given focus and latus rectum, touching a given straight line.

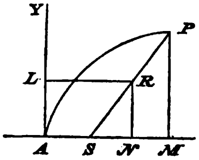
5. Given the focus and position of the axis; describe a parabola touching a straight line given in position.

6. Describe a parabola with a given focus touching two straight lines given in position.

7. Given a straight line and two points on the same side of it; shew that in general two parabolas may be described passing through the points and having the line for directrix. Also shew that when only one parabola can be described, the distance of the points from each other equals the sum of their distances from the directrix.

8. A parabola being traced and its axis given, determine the focus. A method nearly the inverse of that used in Prob. 1, may be used here.

9. The circle described upon the radius vector SP of a parabola touches the tangent at the vertex.



Let SP be the radius vector, Y the tangent at the vertex, bisect SP in R and draw LR perpendicular to AY. Drop perpendiculars PM, RN, on the axis; then, because $SP = 2SR$, by similar triangles SRN, SPN,

$$SM = 2SN,$$

and

$$RL = AN = AS + SN = \frac{AS}{2} + \frac{AS}{2} + \frac{SM}{2}$$

$$= \frac{AS + AM}{2} = \frac{SP}{2},$$

therefore $RL = RS = RP$, and the circle described upon SP, with centre R, will touch AY in L. Q. E. D.

This property is sometimes useful in determining parabolas, as in the following cases.

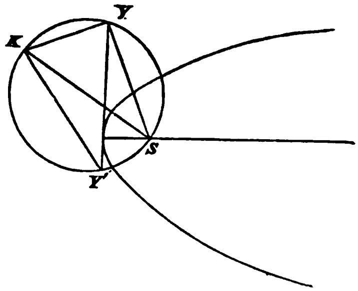
10. Given the focus; determine a parabola passing through a given point and touching a given straight line.

11. Given the focus; determine a parabola touching a given straight line at a given point.

12. Given the focus, a point, and the latus rectum; describe the parabola.

13. Given the focus; describe a parabola passing through two given points.

14. Draw two tangents to a given parabola through a given point.



Let K be the given point; join SK and describe a circle upon it as diameter. Let Y and Y' be the points at which it cuts the tangent at the vertex; join KY, KY', then each of the angles KYS, $KY'S$, in a semicircle, is a right angle. Hence the lines KY, KY' produced will touch the parabola.

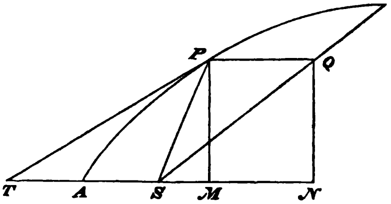
15. The directrix is the locus of the points from which tangents at right angles to each other can be drawn to the parabola.

16. If PYT be a tangent meeting the tangent at the vertex in Y and the directrix in T, then $PY.PT = SP^2$, $PY.YT = AS.SP$.

17. The line drawn from S to the point where the tangent at P meets the directrix is at right angles to SP. Hence shew that tangents at the extremities of a chord through the focus are at right angles to each other.

18. If the tangents at P and Q intersect in T, and PM, QM', TN, be drawn perpendicular to the axis; then $2TN = PM + QM'$, or $PM - QM'$, according as P and Q are on the same or different sides of the axis.

19. Find the locus of the point of intersection of the focal chord with its diameter.



Let Q be the point of intersection, PT the tangent at the vertex of the diameter; then QS is parallel to PT. Draw QN perpendicular to the axis.

Then $\qquad SN = TM = 2AM$ and $PM = QN$;

but $\qquad PM^2 = 4AS.AM$,

therefore $\qquad QN^2 = 2AS.SN$;

or the locus of Q is an ellipse whose vertex is S, and whose latus rectum equals half that of the original parabola.

20. If chords be drawn to a parabola passing through the point of intersection of the directrix with the axis, shew that the locus of their middle points is a parabola whose latus rectum is half that of the original one; also find its focus.

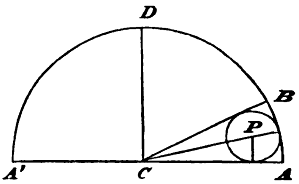
21. If from the focus S perpendiculars be drawn on the normals, find the locus of the points of intersection. It is a parabola whose vertex is at S, and whose latus rectum is one-fourth of that of the original parabola.

22. Find the locus of the foci of all parabolas having a given point for vertex and touching a given straight line.

The locus as a parabola whose axis is perpendicular to the given line, its vertex being at the given point and its latus rectum equal to the distance of the point from the given line.

23. CB is the radius of a circle which revolves from CA to CA'; find the locus of the centre of the circle which is in-

scribed between CA, CB, and the curve. By the property of the directrix it is easily shewn to be a parabola, whose focus is C and whose directrix is the tangent at D.



24. If P be a point in the parabola, A the vertex,

$$AP^2 = AM(AM + 2MN),$$

N being the foot of the normal.

25. If PL be drawn perpendicular to the chord AP, and M be the foot of the ordinate, then ML = latus rectum.

26. If from any point in a parabola, focus S, two lines PF, PH, be drawn making equal angles with the normal PG, then $SG^2 = SF . SH$.

27. If G be the foot of the normal, P a point in the parabola, $PN^2 = L . SP$, $PT^2 = 4 . AM . SP$, T being the intersection of the tangent with the axis.

28. If from a point P in the parabola, PQ be drawn perpendicular to the directrix, and PAS through the vertex to meet it, $SK = QK$, K being the intersection of the directrix and axis.

29. The tangent at any point of the parabola meets the directrix and the latus rectum produced in points equally distant from the focus.

30. If Pp be any chord, PM and pN ordinates at its extremities, O the point in which it cuts the axis, then $AM . AN = AO^2$.

31. If AP, Ap be at right angles to each other, then AM, $AN = L^2$, L being the latus rectum, M and N as above.

32. If from G the foot of the normal, GK be drawn perpendicular to SP, then PK is equal to one-fourth of the latus rectum.

This property, which holds for the ellipse and the hyperbola also, will be found very useful in applications of the third section of Newton's *Principia.* The proof is too easy for a solution to be requisite.

33. If SL be drawn from S perpendicular to the normal, then $SL^2 = Am . SP$, Pm being the ordinate at the point P.

34. If Pn be the normal, $Pn^2 = L . SP$.

35. If PSp be a focal chord, $L . Pp = 4 . SP . Sp$.

For $SP = AM + AS = 2AS + SM,$

$Sp = Am + AS = 2AS - Sm,$

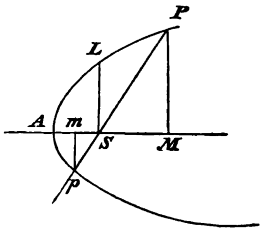
$\therefore SM = SP - SL, \quad Sm = SL - Sp,$

and $SP : Sp :: SM : Sm :: SP - SL : SL - Sp;$

$\therefore SP(SL - Sp) = Sp . (SP - SL),$

$\therefore (SP + Sp) SL = 2 . SP . Sp,$

or $Pp . L = 4 . SP . Sp.$



36. If PG, pg be the normals at P and p, then $PG^2 + pg^2 = L . Pp$.

37. Draw through the focus of a parabola a chord of a given length. The property that the ordinate to any diameter drawn through the focus is the parameter to that diameter will here be found useful.*

38. If a diameter meet the focal chord in N, the parabola in P, and the directrix in K, $PV = PK$.

39. If in the last problem the focal chord meet the directrix in H, $KH^2 = HS . HV$.

40. Given a diameter with its vertex, the parameter and the inclination of the ordinates, determine the parabola.

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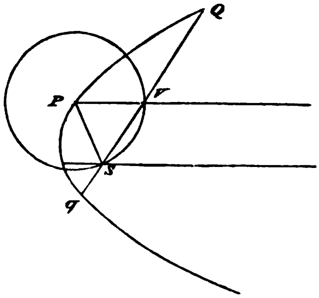
* Dr. Whewell's *Conic Sections*, Prop. VIII.



With P as centre and one-fourth of the parameter for radius, describe a circle cutting the diameter in V, through V draw $QVSq$ making the given angle with PV, QV and qV will be each equal to half the parameter, and therefore Qq a focal chord (see Prob. 37): then, since $PV = PS$, the other point in which $QVSq$ cuts the circle will be the focus, and the parabola is easily determined.

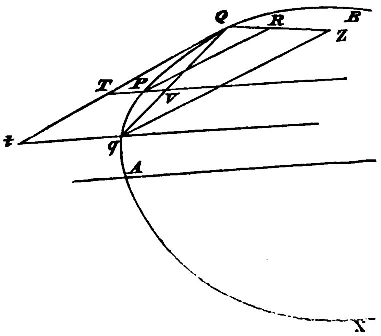
41. Draw a tangent to a parabola making a given angle with the axis.

42. A parabola being drawn, find its axis.

43. A parabola being drawn, find its focus.

44. If two equal parabolas have a common axis, a straight line touching the interior parabola, and terminated by the exterior, will be bisected by the point of contact.

45. If Qq be an ordinate to a diameter PV, and QT the tangent at Q meet the diameter in T, then $PV = PT$.



Draw QTt the tangent at Q, QRZ the diameter at Q, and tq parallel to TP, through P and q draw PR and qZ both parallel to QTt, then PR and qZ are ordinates to the diameter.

$$QRZ : PR^2 : QZ^2 :: QR : QZ,$$

or $\qquad QT^2 : Qt^2 :: PT : qt \;\ldots\ldots\ldots\ldots (1);$

but $\qquad QV : Qq :: QT : Qt :: TV : qt,$

therefore $\qquad 1 : 2 :: QT : Qt :: TV : qt \ldots\ldots\ldots (2),$

therefore by (1) $\qquad 1 : 4 :: PT : qt;$

also by (3) $1 : 4 :: TV : 2qt :: \frac{TV}{2} : qt,$

therefore $\frac{TV}{2} = TP.$

46. If TP be a tangent at P, and a chord PP' be drawn perpendicular to it, then if $P'T$ be a tangent at P', PT will be bisected in the directrix.

The property proved in Prob. 14, and that just demonstrated, will be of use here.

47. Inscribe in a parabola given in position, a triangle whose sides shall be parallel to three given lines, none of which are parallel to the axis of the parabola.

48. If a right-angled triangle be inscribed in a parabola having its base perpendicular to the axis and its vertex at the vertex of the parabola, its area equals L^2.

49. In a parabola, if tangents be drawn at two points, the angle between the tangents equals half the angle between the radii vectores.

50. If CC' be any chord in a parabola, S the focus, and SY the perpendicular upon the tangent at its middle point, then $$SY.(SC' - SC) = C'C.AY.$$

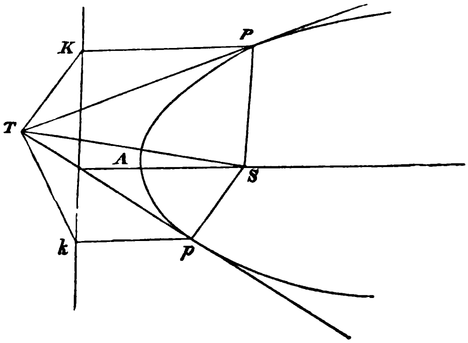
51. If PM be the middle point of CC' a chord in the parabola, and MQ be drawn perpendicular to CC' to intersect SP in Q, prove that $SQ = \frac{SC + SC'}{2}$.

52. Describe a parabola such that the angle made by the tangent with a radius vector of given length may equal a given angle.

53. If Q be any point in an ordinate to the axis, QP a diameter, PP' a focal chord, $P'Q'$ a diameter meeting the same ordinate in Q', then $QP + PP' + P'Q'$ is the same for all positions of Q in the same ordinate.

54. The straight line joining the focus of a parabola with the intersection of two tangents, makes equal angles with straight lines drawn from the focus to the points of contact.

Drop PK, pk perpendicular on the directrix, and join TK, Tk,



because $PK = PS$, $\angle KPT = \angle TPS$,

and side PT is common, therefore $\triangle PKT = \triangle PST$,

therefore $\angle PKT = \angle PST$ and $TK = TS$.

Similarly $\angle pkT = \angle pST$ are $Tk = TS$,

therefore $TK = Tk$, $\therefore \angle TKk = \angle TkK$,

therefore $\angle PKT = \angle pkT$,

therefore $\angle PST = \angle pST$.

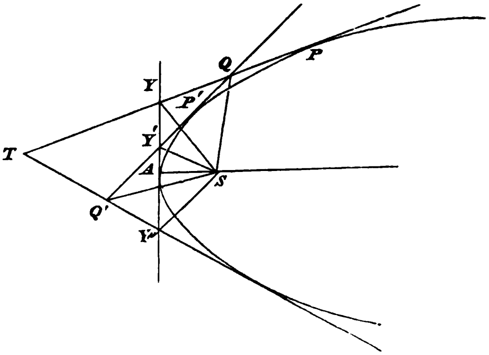
Wallace's *Conic Sections*, Prop. v.

55. If a straight line be drawn from the intersection of any two tangents to the focus and another perpendicular to the directrix, these will make equal angles with the tangents.

Ibid. Prop. iv.

56. If tangents to a parabola at P and P' intersect in Q, then, whatever be the position of $P'Q$, $P'QS$ is an invariable angle.

Draw $AY'Y$ the tangent at the vertex, and draw SY and SY' perpendiculars on the tangents. Then, because the angles at Y and Y' are right angles, therefore a circle may be described about $SY'YQ$; therefore the angle $Y'QS$ equals the angle $Y'YS$, which remains fixed while YP remains fixed.



57. The circle passing through the intersections of three tangents to a parabola, passes also through the focus.

As in the last problem, in the triangles SYY', SQQ',

$$\angle SQY' = \angle SYY', \quad \angle SQ'Y' = \angle SY''Y';$$

therefore the remaining angle QSQ' equals the angle YSY', which, together with YTY', equals two right angles, because the angles at YY'' are right angles; therefore QSQ', together with QTQ', make up two right angles. Hence $QSQ'T$ is inscribable in a circle, or the circle which passes through QT and Q' passes also through S.

From this problem, combined with a common property of the parabola, we can deduce the following geometrical theorem.

If any triangle be inscribed in a circle, and from any point in the circumference perpendiculars be drawn to its sides, or its sides produced, the points of intersection will all lie in one straight line.

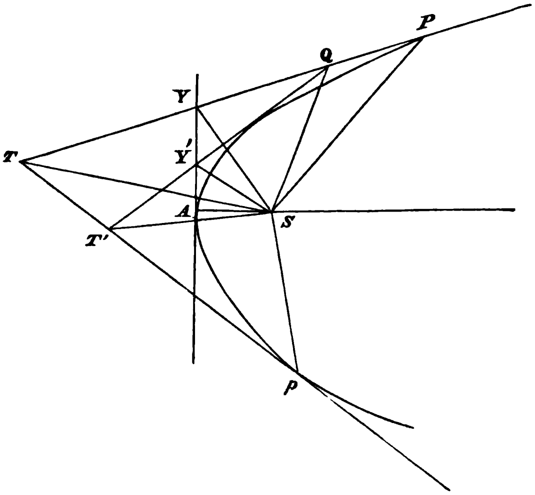
For we see that the point may be the focus of a parabola to which the three sides of the triangle are tangents, and the perpendiculars drawn from S to them must intersect them in the tangent at the vertex of the parabola.

58. Describe a parabola touching four given straight lines.

This may be done by means of the property just proved. Hence also we may infer, that the circles described about the four triangles formed by four lines which intersect, all pass through the same point.

59. If SP.Sp be drawn to the points of contact of two tangents which intersect in T, the triangles STP, STp are equiangular.

Draw QT' any other tangent cutting PT and pT in Q and T'.



Then, because a circle may be described about $SY'YQ$,

$$\therefore \angle SYY' = \angle SQY';$$

but $$\angle SYY' = \angle SPY \text{ by similar triangles,}$$

$$\therefore \angle SPY = \angle SQY':$$

and because a circle may be described about $QTT'S$, therefore the angles SQT', STT' in the same segment are equal;

$$\therefore \angle SPT = \angle STT'.$$

And we know from Prob. 54, that the angle PST equals the angle pST; therefore the triangles PST, pST are equiangular.

60. If tangents at P and p intersect in T, $SP\,Sp = ST^2$.

61. The triangles PTS, pTS, $QT'S$ are similar.

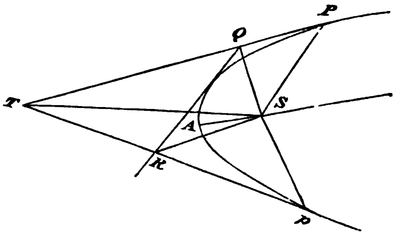
62. PT, pT being tangents at P and p,

$$PT^2 : pT^2 :: SP : Sp.$$

63. If PT, pT be two tangents to a parabola, and they be cut by a third tangent in Q and R, then

$$TQ : pR :: QP : RT.$$

Join QS, RS, ST: then, since T, Q, S, R are points in a circle (Prob. 57),



therefore $\angle TQS + \angle TRS = 2$ right angles;
but $\angle TQS + \angle SQP = 2$ right angles,
$\therefore \angle PQS = \angle TRS.$

Also (Prob. 59) the angle TPS equals the angle STp; therefore the triangles SQP, STR are similar. In like manner the triangles TQS, RSp may be shewn to be similar.

Hence $QP : QS :: TR : RS$,
or $QP : TR :: QS : TS$.
Similarly, $TQ : Rp :: QS : TS$,
therefore $QP : TR :: TQ : Rp$,

or the alternate segments of the tangents are proportionals.

64. If two equal tangents be drawn to a parabola, the alternate segments of them made by a third tangent will be equal.

65. If two tangents be drawn at the extremities of a chord, and a third tangent be drawn parallel to the chord, the portion of this, which is intercepted by the other tangents, is bisected at the point of contact.

66. If P, p, p_1 be points of contact of three tangents to a parabola, T, R and Q the points of their intersection; then $SQ.SR.ST = SP.Sp.Sp_1$.

This may be easily demonstrated by the property proved in 59.

67. If R be the radius of the circle described through the three points of intersection of tangents at points P, p, p_1 in a parabola, latus rectum L, then

$$R^2, L = SP.Sp.Sp_1.$$

This is easily proved by means of *Euclid*, Prop. C., Book VI., and the property of Prob. 57.

68. $SY.SY'.SY''$ being the perpendiculars on the tangents at three points, R the same as above,

$$RL^2 = 8.SY.SY'.SY''.$$

The results of the last three problems are proved analytically in *The Mathematician*, No. 4.

69. The radius of the circle described passing through the vertex of a parabola, and the extremities of the latus rectum $= 5AS$.

70. If a circle be inscribed in a portion of a parabola cut off by an ordinate $2Pm$, the radius of the circle will equal $Pm - 2AS$.

71. If a circle be circumscribed about a portion of a parabola cut off by an ordinate $2Pm$, the diameter will equal $L + Am$.

72. Describe a parabola with a given vertex, touching a given straight line in a given point.

If, P being the given point, A the vertex, PA be produced to P', so that $AP' = AP$, and a circle be described on AP, then drawing through p' a line parallel to the given line, cutting the circle in m or m', it may be shewn that the parabola required will have Am or $A'm$ for its axis.

73. Describe a parabola touching a given circle at a given point, and having its axis coincident with a given diameter of the circle.

74. Two equal parabolas having the same focus and their axes turned contrary ways, intersect at right angles.

75. Draw a parabola intersecting a given parabola at right angles at the extremities of a given ordinate.

76. If a parabola roll on another equal parabola, the focus traces out the directrix.

77. If PT be a tangent at P, Qq a chord parallel to it, and QN, qn, PM, pm ordinates to any diameter, then $QN + qn = 2PM$, or $QN - qn = 2Pm$, according as Q and q are on the same or different sides of the diameter.

Emerson's *Conic Sections*, Prop. 45.

78. If AT be the axis, AYE and PT tangents at A and P, PE being the diameter at E, then if AE and PT intersect in Y, the triangle AYT equals the triangle PYE.

Ibid., Prop. 15.

79. If QqR be a chord parallel to the tangent at P, meeting the axis in R, and the tangent AE at the vertex intersect the diameter at P in E, then QN being the ordinate to the axis at Q meeting the diameter at P in F, the triangle QNR equals the rectangle $AEFN$.

80. If any chord PQ of a parabola be produced to meet the directrix in O, and PEH be drawn through the focus, then OF bisects the angle QEH.

Emerson's *Conic Sections*, Prop. 35.

81. Describe a parabola with a given focus passing through three given points. Joining one of the points with the focus

and the other two, we easily determine the directrix by means of the last problem.

82. BC CD are consecutive arcs of a parabola, and the lines drawn from M and N the middle points of the chords, parallel to the axis to meet the parabola in P and Q, are equal; prove that the chord BD is parallel to the tangent at C.

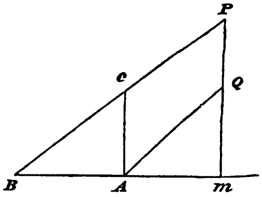
If a tangent be drawn to the parabola at C, and Bb, Dd be drawn parallel to the axis to meet it in b and d, and MP and NQ be produced to meet it in m and n, it may be proved that $Bb = Dd$.

83. If AB, BC be two consecutive arcs of a parabola, and LW, GH, MV, the portions of the diameters to the chords AB, AC, BC, intercepted between the chords and the curve, prove that

$$\sqrt{(LW)} + \sqrt{(MV)} = \sqrt{(GH)}.$$

84. Find the locus of the centre of a circle touching a given circle and a given straight line. The locus is a parabola.

85. AC is drawn at right angles to BAM, and is made equal to BA, PM is an ordinate to the line BCP, and AQ is drawn to a point Q in PM so that AQ may equal PM. Find the locus of Q. Answer, a parabola.



86. If from the foot of the directrix, a chord be drawn meeting the parabola in P and SL in K, and SP be produced meeting RQ parallel to the axis in Q, then Q describes a circle about S whose radius is $\dfrac{L}{2}$.

87. If AL be the line drawn through the vertex of a parabolic section of a right cone in a plane at right angles to that of the section, and AR be drawn perpendicular to BC,

the other side of the triangular section made by this plane, then $2RL$ equals the latus rectum of the section.

88. If F be the focus of a parabolic section, and AG be taken in AB equal to AF, then a line through G, parallel to AL, intersects the plane of the parabolic section in the directrix.

89. In the parabola the diameter of curvature

$$= \frac{(\text{normal})^3}{\left(\dfrac{(\text{latus rectum})}{2}\right)^2}.$$

This important property, which is easily proved from the ordinary expression $\left(\text{viz. } \dfrac{4SP^2}{SY}\right)$ for the diameter of curvature, will be shewn to hold for the ellipse and hyperbola also.

90. The radius of curvature at the extremity of the latus rectum equals twice the normal.

91. The circle of curvature at a point P in a parabola, cuts off from the diameter at P a portion equal to the parameter of that diameter.

92. The chord of curvature at a point P in the parabola passing through the vertex A equals $\dfrac{4PY^2}{AP}$.

If PO be the radius of curvature at P, and DK be drawn perpendicular to AP; PK will be half the chord of curvature through the vertex: also if AZ be drawn perpendicular to TP, then from similar triangles POK, APZ,

$$PK : PO :: AZ : AP;$$

by similar triangles AZT, SYP,

$$SP : SY :: AT : AZ;$$

therefore $$SP.PK : PO.SY :: AT : AP,$$

and $$PO = \frac{2SP^2}{SY} AT = AM,$$

M being the foot of the ordinate at P;

therefore $$PK : 2SP :: AM : AP;$$

and, by similar triangles, SYP, PYR. (YR being perpendicular to PM),
$$SP : PY :: PY : AM,$$

therefore $$PK . AP = 2PY^2;$$

therefore $2PR$ = chord of curvature through the vertex
$$= \frac{4PY^2}{AP}.$$

93. The circle of curvature cuts off from the axis of the parabola a magnitude $= \frac{4 . SP^2}{SA}$.

94. If R and R' be the radii of curvature at the extremities of a focal chord,
$$\frac{1}{R^{\frac{2}{3}}} + \frac{1}{R'^{\frac{2}{3}}} = \left(\frac{2}{L}\right)^{\frac{2}{3}}.$$

See Prob. 35.

95. If n and n' be the normals at the extremities of a focal chord, then
$$\frac{1}{n^2} + \frac{1}{n'^2} = \left(\frac{2}{L}\right)^2.$$

96. If PM be the ordinate at a point P, and T be the intersection of the tangent at P with the axis,
$$TP . TY = TM . TS.$$

By similar triangles $TYS . PTN$,
$$TP . TM :: TS : TY.$$

Senate-House Papers, 1848.

97. PSp is a focal chord of a parabola, RD is the directrix meeting the axis in D, Q is any point in the curve; prove that if QP, Qp produced meet the directrix in R, r, half the latus rectum will be a mean proportional between DR, Dr.

The solution of this problem follows immediately from the property of Problem 80; viz.

If any chord PQ of a parabola be produced to meet the directrix in O, and PSp be drawn through the focus S, then OS bisects the angle QSp. The solution is as follows.

Draw Qn parallel to OS, meeting PS in n, and draw PK and Qk perpendicular to the directrix; then, by similar triangles,

$$PK : QK :: PO : QO :: PS : nS;$$

but $$PK = PS, \quad \therefore QK = nS = QS;$$

therefore $$\angle SnQ = \angle SQn;$$

therefore, by parallel lines,

$$\angle OSp = \angle OSQ.$$

In the problem before us, joining RS, rS, the $\angle RSQ + \angle rSQ = \frac{1}{2}$ two right angles, equals a right angle; therefore a semicircle described upon Rr will pass through S; therefore

$$RD.Dr = DS^2 = \left(\frac{L}{2}\right)^2.$$

Q. E. D.

It is easily shewn that R and r will always fall on different sides of D, and therefore this construction will always apply.

We may observe that when Q is in the line drawn through p perpendicular to the axis, Dr is infinite, and the formula would appear to fail, but we find easily that in this case PQ produced will always pass through D, and thus $DR.Dr = o\,\infty$, which is of course indeterminate.

Senate-House Papers, 1848.

98. PSp is any focal chord of a parabola whose vertex is A; prove that AP, Ap will meet the latus rectum in two points Q, q, whose distances from the foci are equal to the ordinates of the points P and p respectively.

This is $$QS : qS :: pm : PM.$$

The property $Am.An = AS^2$ (Prob. 30), which is easily proved, is here useful. From the similar triangles Apm, AqS, AQS, APM, we get

$$Am : AS :: pm : QS,$$
$$AS : AM :: QS : PM;$$

whence, with the aid of the property to which we have referred, the solution is easy.

Senate-House Papers, 1848.

99. If from any point P in a parabola a double ordinate Pp be drawn to the major axis and a tangent PRT, and any diameter FQR cut the ordinate in F, the curve in Q, and the tangent in R; then $RQ : QF :: PF : pF$.

This property may be extended to the case of any ordinate.

100. P is a point in a parabola, PG the normal; if GQ be drawn to meet PS produced in Q, and QO perpendicular to PQ to meet PG produced in O, O is the centre of the circle of curvature at P.

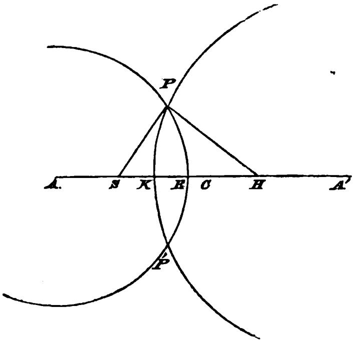
THE ELLIPSE.

An Ellipse may in general be described to fulfil five conditions; or three, if the position of the vertex, focus, or centre be assigned. We shall consider an ellipse *described* when the two foci, or a focus and the directrix, are determined, together with the magnitude of the major axis.

By the definition of an ellipse, the distances of any point in the curve from the focus and the directrix are in a given ratio, which is called the *eccentricity:* this is sometimes expressed by saying that they are as $e : 1$, where e is less than 1.

1. To find geometrically any number of points in an ellipse when the magnitude of the major axis and the position of the foci are given.

Let S and H be the foci. Bisect SH in C, and produce CS, CH to A and A', making $CA = CA' = \frac{1}{2}$ major axis; with centre S and any radius SE greater than SA, describe a circle; take $AK = SE$, and with centre H and radius $A'K$ describe a circle cutting that whose centre is S, in P and P'; P and P' may easily be shewn to be points in the ellipse, and any number of others may be similarly determined.



If SPH be a string of length equal to the axis major, fastened at S and H, a pencil, so moved as to keep the string stretched, will trace out the ellipse.

2. Describe an ellipse of which the focus and the position and magnitude of the major axis are given.

3. Describe an ellipse with a given focus touching *three* given straight lines, no two of which are parallel and on the same side of the focus.

Draw SY, SY', SY'' perpendiculars upon the three tangents; then, since Y, Y', Y'' lie in the circle described upon the major axis, the circle circumscribed about the triangle $YY'Y''$ will have its diameters equal to the major axis, and its centre will be also that of the ellipse. The remainder of the construction is plain.

4. Given two points, the focus and the eccentricity; describe the ellipse.

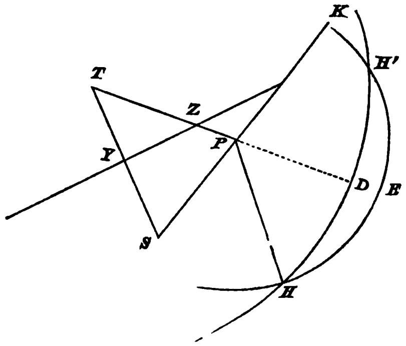
If P, P' be the points, S the focus; then, if a tangent be drawn to the two circles described with radii, which are to SP, SP' as $e : 1$, a tangent to these circles will be the directrix.

5. Given two points, the directrix and the eccentricity; describe the ellipse.

6. Given the foci; describe an ellipse touching a given straight line.

7. Given the focus and the length of the major axis; describe an ellipse touching a given straight line and passing through a given point.

Let fall SY a perpendicular upon the given straight line, make $YT = SY$, and from T, with a radius equal to the given axis major, describe HDH'. Produce SP to K so that SK may equal the given axis major; then with centre P and radius PK describe a circle HEH'. An ellipse described with foci S, H, or S, H', will pass through P, and will touch YZ in points which may be determined by joining TH, TH'. P must lie on the same side of the given line as the focus, otherwise



SP would be greater than TP, and therefore PH, which = major axis − SP, would be less than PD, which = major axis − TP.

8. Given the focus and the length of the major axis; describe an ellipse passing through two given points.

9. Given the focus and the length of the major axis; describe an ellipse touching two given straight lines.

10. Given a focus; describe an ellipse touching two given straight lines in two given points, and state how the positions of the points and lines are restricted.

11. Find the locus of the centres of all the ellipses having the same focus and touching a given straight line.

12. Why was the "latus rectum" of a Conic Section so called? Prove the property from which the Ellipse derives its name.

The ancient geometricians found that in the three species of Conic Sections, the square of the ordinate was respectively less or greater than the rectangle, one side of which was the ordinate through the focus, and the other the abscissa of the point measured from the vertex, or otherwise equal to it. Hence this ordinate was called the latus rectum; and the curves, the ellipse, the hyperbola, and the parabola.

In the case of the ellipse, let Pm be any ordinate, Am the abscissa measured from the vertex; then

$$Pm^2 : Am \, . \, mA' :: Bc^2 : Ac^2,$$

$$Pm^2 : Am \, . \, \{2Ac - Am\} :: Bc^2 : Ac^2 \, \ldots\ldots \, (1).$$

Take a point n in AA' such that

$$An : Am :: Am : 2Ac.$$

Then (1) becomes

$$Pm^2 : 2Ac \, . \, \{Am - An\} :: Bc^2 : Ac^2.$$

But L being the latus rectum,

$$2Ac : L :: Ac^2 : Bc^2;$$

therefore $$Pm^2 = L(Am - An),$$

or Pm^2 is greater than $L \, . \, Am$.

13. If K be the point where the directrix intercepts the axis major produced, then

$$Ac : Sc :: AK : AS, \text{ or } Sc = e.Ac.$$

The property $SB = AC$ is useful in this case.

14. If the tangent at a point, whose abscissa measured from the vertex is AM, meet the axis major produced in T, AM, AA', AT will be in harmonic progression.

15. If P be any point in the ellipse, and the chords AP, $A'P$ be produced to intersect a perpendicular to the axis through T (the point where the tangent at P meets the axis) in Q and Q', then $QT = Q'T$.

16. G being the foot of the normal, $CG \, . \, CT = SC^2$.

17. If any number of ellipses be described upon the same major axis, tangents at the extremities of the latera recta all pass through the same point.

18. If AP, $A'P$ be chords drawn from the vertices to a point P in the ellipse, and PK, PK' be perpendiculars to them, the line KK' intercepted on the axis equals the latus rectum.

19. If PM be an ordinate, meeting the circle on the major axis in P', PG a normal at P, and BD a perpendicular from the extremity of the axis minor upon CP', then the triangle PGM = triangle BCD.

20. If PQP' be a chord of the circle described on the major axis of an ellipse, and another circle be inscribed on the minor axis cutting the chord in Q and Q', then $PQ.P'Q$ equals SC^2.

21. Tangents drawn to an ellipse from any point in the circumference of the circle described on the axis major, include an obtuse angle.

22. S and H being the foci, CA and CB the semi-axes, prove that when the angle SBH is a right angle,

$$CA^2 : CB^2 :: 2 : 1.$$

23. Prove that any parabola described with centre of the above ellipse for vertex, will cut the ellipse at right angles. This is easily proved by the property of the sub-normal.

24. If an ellipse and parabola have the same vertex and focus A and S, then if A' be the other extremity of the major axis of the ellipse,

$2AC : A'S ::$ lat. rect. of parabola :: lat. rect. of ellipse.

25. If they have the same foci and latera recta, then, O being the vertex of the parabola,

$$2AC : A'S :: AS : SO.$$

26. Given the lengths of the axes of an ellipse, and the positions of one focus, and of one point in the curve; give a geometrical construction for finding the centre. See Prob. 7.

Senate-House Papers, 1848.

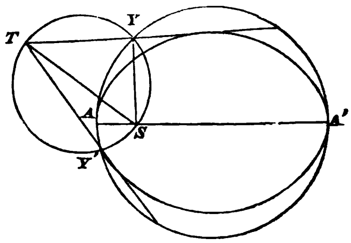
27. An ellipse being traced, find its major axis and foci. Draw any two parallel chords and bisect them, produce the line joining the bisections to meet the ellipse, its middle point will be the centre; about the centre describe any circle cutting the ellipse, join the points of intersection, then the

diameters bisecting the chords will be the axes, and a circle described from the extremity of the minor axis with radius equal to the semi-axis major, will cut the major axis in the foci.

28. Draw a tangent to the ellipse inclined at a given angle to the major axis.

29. Draw two tangents to an ellipse from a given point without it.

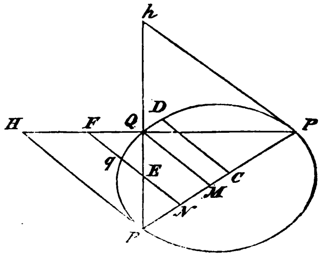
Let T be the given point, S the focus of the given ellipse; on AA' the major axis describe a circle, join ST, and on it as diameter describe another circle cutting the former in Y and Y'; join TY and TY'; then these lines are intersected by perpendiculars from the focus in the circumference of the circle on the major axis, the essential property of tangents to the ellipse; these lines are therefore tangents.



30. The circle described on the radius vector SP of an ellipse, touches the circle described on the major axis. Prove this, and hence draw a tangent to the ellipse at a given point. This property reduces the problem of describing an ellipse with a given focus through three given points, to that of describing a circle touching three given circles, and so with other cases.

31. If Pp be any diameter, and if through any point in the curve as Q, two lines PQH, pQh be drawn, and also the ordinate qN intersecting the lines in E and F, then NE, Nq, NF are in continued proportion.

Draw CD conjugate to CP, and tangents $PH.Ph$; then, by similar triangles,



$$Pp : Ph :: pM : QM,$$

$$Pp : Hp :: PM : QM;$$

therefore $Pp^2 : Ph . Hp :: PM . pM : QM^2$

$$:: PC^2 : CD^2 :: (2PC)^2 : (2CD)^2;$$

therefore $PhHp = (2CD)^2.$

Also $Pp^2 : pM . PM :: (2CD)^2 : QM^2,$

and by similar triangles $Pp : pN :: Pk : NE,$

$$Pp : PN :: Hp : NF;$$

therefore $Pp^2 : PN . pN :: PH . Hp : NE . NF;$

but $Pp^2 : PN . pN :: (2CD)^2 : qN^2,$

therefore $qN^2 = NE . NF.$ Q. E. D.

32. *SL* is the semi-latus-rectum, *A* the vertex, *LA* produced meets the directrix in *Q*, and *QS* intersects the tangent at the vertex in *R*; prove that $AR = AS$.

33. If a tangent be drawn at any point *P*, intersected by the tangents at *A* and *A'* in *H* and *K*, and by the minor axis produced in *q*, then $HA : Cq :: PM : A'K$.

34. *AH* and *A'K* being as above, *SY* and *HE* perpendiculars on any tangent, $AH : SY :: HE : A'K$.

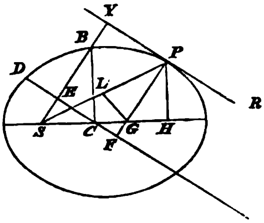
35. A circle described upon *HK* passes through the foci.

36. Describe an ellipse on a given line as axis major, touching a given straight line.

37. Inscribe an ellipse in a given parallelogram.

38. If a diameter be drawn through *L* the extremity of the latus rectum, to intersect the directrix in *O*, and from the other point *p*, where it meets the curve, a line *pSK* be drawn, then the angle *OSL* equals the angle *OSF*.

39. If from G the foot of the normal at P, GL be drawn perpendicular to SP, PL equal $\dfrac{\text{latus rectum}}{2}$.



We know that $PG : BC :: BC : PF$,* and by similar triangles, PGL, PEF,

$$PL : PG :: PF : PE;$$

therefore $PL : BC :: BC : PE$,

or $PL : BC :: BC : AC \because PE = AC$.

Hence PL equals $\dfrac{\text{latus rectum}}{2}$.

The solution of this problem has been given, although it is found in Hustler's *Conic Sections*, Prop. XI., Cor., because this property is very useful in applications of the 3rd Section of Newton, as is the following construction.

40. Given the focus, the tangent at a given point, and the magnitude of the latus rectum; construct the ellipse.

Draw a line perpendicular to the tangent at the given point (see fig. of last Problem) in SP, take PL equal $\dfrac{\text{latus rectum}}{2}$, and draw LG at right angles to SP intersecting the normal in G, join SG and produce it indefinitely; it will be the direction of the major axis; and making the angle HPR equal SPY, the line PH will intersect it in the other focus H; bisect $\angle SH$ in C, drop SY a perpendicular upon the tangent, and describe a circle with radius CY which will cut SH produced in the extremities of the major axis, and the ellipse is determined.

41. If the normal cut the major and minor axes in G and G', $PG'.PF = AC^2$, $PG.PG' = CD^2$.

42. The sum of the squares of the normals at the extremities of conjugate diameters is constant.

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* Dr. Whewell's *Conic Sections*, Prop. XVIII.

43. PG being the normal at P, CD the conjugate to CP,

$$CD^2 - PG^2 = SG.GH.$$

44. The letters being as usual, GL perpendicular upon SP,

$$SG : SP :: SC : AC :: GL : PM.$$

This property, which is easily proved by Euclid, Book VI. Prop. 3, will be found useful in problems in central forces.

45. Prove that $2PG : L :: SP : SY$, L being the latus rectum.

46. Inscribe a circle in the portion of an ellipse cut off by the latus rectum.

47. If PG' be the normal at P cutting the axis minor in G'; then G', S, and H all lie in the circumference of a circle whose centre is on BC.

48. If a length $PQ = CD$ be taken in the normal to an ellipse, the locus of the point Q is a circle whose radius is $AC - BC$.

49. If P and D be the extremities of conjugate diameters,

$$SP.HP + SD.HD = AC^2 + BC^2.$$

50. $$(SP - AC)^2 + (SD - AC)^2 = SC^2.$$

51. If P and D be the extremities of conjugate diameters, PM and DN ordinates to the major axis,

$$PM^2 + DN^2 = BC^2. \; CM^2 + CN^2 = AC^2.$$

52. If PSp be any focal chord, L the latus rectum,

$$L.Pp. = 4SP.Sp.$$

In many problems the following property of the ellipse (Hustler's *Conic Sections*, Prop. XVIII. Cor. 1) will be found very useful. If any two chords Pp, Qq intersect in F, and CD, CD' be semidiameters parallel to them, then

$$PF.Fp :: QF.Fq :: CD^2 : CD'^2,$$

which relation holds wherever the point F may be, provided the chords remain parallel.

53. PSp, QCq are any two parallel chords drawn through the focus and centre of an ellipse; prove that

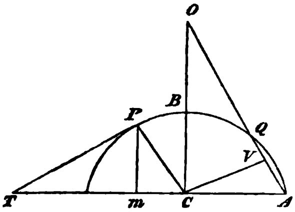
$$SP.Sp : CQ.Cq :: BC^2 : AC^2.$$

54. P and D being the extremities of semiconjugate diameters, PSp, DHd focal chords, SL the semi-latus-rectum,

$$SP.Sp + HD.Hd = BC^2 + SL^2.$$

55. If Pp be any focal chord, CD a diameter parallel to Pp; then $Ac.Pp = 2CD^2$.

56. If a chord AQ drawn from the vertex be produced to meet the minor axis in O, and CP be a semidiameter parallel to it, then $AQ.AO = 2CP^2$.



Draw a tangent at P meeting the axis in T. Bisect AV in Q, join CV; then, because QA is an ordinate to CV, and is parallel to CP, therefore CV is parallel to PT. Hence, by similar triangles ACP, PTM,

$$AV : CA :: CP : CT,$$

also
$$AO : CA :: CP : CM;$$

by similar triangles AOC, PMC;

therefore $$AV.AO : CA^2 :: CP^2 : CM.CT;$$

therefore, since $CA^2 = CM.CT$, $AQ.AO = 2CP^2$.

57. If from the extremities of any diameter chords be drawn to any point in the ellipse, diameters parallel to these chords will be conjugate.

Such chords are called supplemental chords.

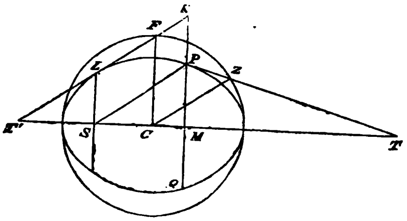
58. Draw a pair of conjugate diameters containing a given angle.

59. Draw a diameter conjugate to a given diameter.

60. If from A and A', the vertices of the major axis, any chords $A'Q$, AQ be drawn to an ellipse cutting the minor axis or the minor axis produced in O and O', then

$$AO.AQ + A'O'.A'Q = 2.(AC^2 + BC^2).$$

61. If from the extremity of the latus rectum a tangent be drawn, and from any point K in it a line $KPMQ$ perpendicular to the axis, meeting the curve in P and Q and the axis in M,



then $SM^2 = KP.KQ$. Join SP, draw PT a tangent at P, and draw CZ parallel to SP; CZ will, by a common property of the ellipse, intersect PT in the circle on the major axis; for the same reason the tangent at L will intersect the same circle in the minor axis produced; then, since $CM . CT = CS . CT'$ (for both equal CA^2);

therefore $CM : CT' :: CS : CT$,

therefore $MT' : CT' :: ST : CT$;

or $MK : CF :: SP : CZ$ by similar triangles,

and $CF = CZ$, $\therefore MK = SP$.

This is a useful property; it is proved in Hustler's *Conic Sections*, Prop. x.

And $SP^2 = PM^2 + SM^2$,

also $KP . KQ = (MK + MP)(MK - MP)$

$= MK^2 - MP^2$.

And from above, $SM^2 = SP^2 - MP^2$,

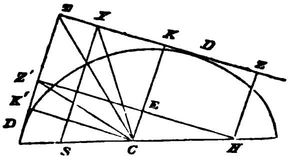
therefore $SM^2 = KP . KQ$.

62. P being any point in the ellipse, $CP^2 - BC^2 = (SP - AC)^2$.

63. If any number of concentric similar ellipses be described on the same line as axis, cutting a line perpendicular to the axis in P', P'', P''', &c., then if CD', CD'', CD''' be conjugate to CP', CP'', &c., D', D'', D''', &c., lie in a line parallel to the axis.

In similar ellipses the ratio of the axes is the same.

64. The locus of the intersection of tangents at right angles to one another in an ellipse is a circle whose (radius)^2 equals $AC^2 + BC^2$.



Let T be the point of intersection of the tangents. Drop CK, CK' perpendiculars upon them from C, HZ, SY, HZ' on them from the foci, join CY, CY'. (CY' is not drawn.)

Then $CY^2 = YK^2 + CK^2$, and $YK = KL = HZ$, also $CY = AC$,

therefore $$AC^2 = CK^2 + HZ^2;$$

similarly, $$CY'^2 = CK'^2 + CE^2 = AC^2,$$

therefore $$2AC^2 = CK^2 + CK'^2 + HE^2 + CE^2$$
$$= CK^2 + CK'^2 + CH^2,$$

because the angle at E is a right angle.

Therefore $$AC^2 + AC^2 - CH^2 = CK^2 + CK'^2,$$

or $AC^2 + BC^2 = CT^2$, which proves the proposition.

65. The construction remaining as above,
$$TY . TZ = TY' TZ' = BC^2.$$

66. If CP, CQ be semidiameters at right angles to one another, then
$$\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{AC^2} + \frac{1}{BC^2},$$

D and D' being the points of contact in 64, the conjugates to CD and CD' will be at right angles to each other; hence from the values of CK and CK' the problem is easily proved.

67. If Pp, Qq be focal chords at right angles to one another, then

$$\frac{1}{Pp} + \frac{1}{Qq} = 2AC.\left\{\frac{1}{AC^2} + \frac{1}{BC^2}\right\}.$$

68. If CP, CQ be at right angles, and CN be drawn perpendicular to PQ,

$$\frac{1}{CN^2} = \frac{1}{AC^2} + \frac{1}{BC^2}.$$

69. Y and Z being the intersections of perpendiculars from the foci upon a tangent at P, the $\angle YAZ = \frac{1}{2}\angle SPH$.

70. If a tangent at a point P meet the major axis in T, and the perpendiculars from the focus and centre in Y and Z, then

$$TY^2 : PY^2 :: TZ : PZ.$$

71. If CP be any diameter, PT intersect the axis in T, and the tangent at A meet CP produced in E, then the triangle ACE equals the triangle PCT.

72. The points E and T in the last problem lie in a line parallel to AP.

73. An ellipse slides between two lines at right angles to each other; find the locus of its centre.

Ans. A circle $(\text{rad.})^2 = AC^2 + BC^2$.

74. If a tangent be drawn at the extremity of the major axis meeting two equal conjugate diameters CP, CD produced in T and t; then $PD^2 = 2AT^2$.

Equal conjugate diameters bisect the chords joining the extremities of the principal axes.

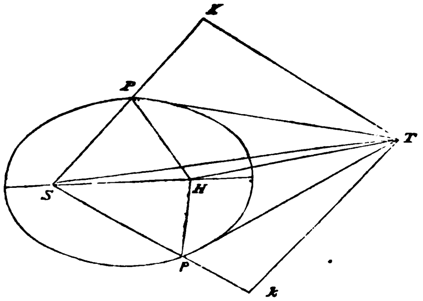
Hustler's *Conic Sections*, Prop. 19.

75. If the tangent at the vertex as above cut any two conjugate diameters in T and t, then $AT.At = BC^2$.

The property $PM : CN :: BC : AC$, PM, DN, being ordinates at P and D, will be useful here.

By means of the property $CP^2 = CV.CT'$, where CV is the abscissa, measured along a diameter CP of the point, the tangent at which intersects CP produced in T', the properties proved in the last five problems may be extended to the case of a tangent at any point, in place of the vertex.

76. If tangents at any points P, p, intersect in T, then the $\angle PST = \angle pST$.



Join ST, HT; produce SP to K and Sp to k, making $PK = PH$. $Pk = pH$. Then in the triangles KPT, HPT the angles at P are equal, PT is common, and $PK = PH$, therefore $KT = HT$; similarly $kT = HT$, therefore $KT = kT$; hence the triangles SKT, SkT have all their sides equal, therefore the $\angle PST = \angle pST$.

Hence a line drawn from the focus to the intersection of two tangents at the extremities of a focal chord is at right angles to that chord.

77. The lines drawn from the intersection of two tangents to the foci make equal angles with the tangents.

78. If PT, pT be any two tangents, and HQk a third tangent intercepted by them, the angle subtended at the focus by HQh will be the same wherever the point of contact Q be taken between P and p.

79. If K be the point where the directrix meets the axis major produced, $CK : SK :: CA^2 : CB^2$.

80. If from the focus S perpendiculars be drawn upon conjugate diameters CP, CD, these perpendiculars produced

backwards will intersect CD and CP in the directrix. The property of 79 is useful.

The property that the rectangles contained by chords intersecting one another in an ellipse are proportional to the squares of the semidiameters parallel to them, extends to the case where the intersection is without the ellipse; hence the rectangles contained by the segments of secants, and therefore the squares of tangents are in the same proportion.

Hustler: Cor. 2, Prop. 18.

This property is useful in the following problems.

81. If Pp, Qq be focal chords, TR, Tr tangents parallel to them,

$$TR^2 : Tr^2 :: Pp : Qq.$$

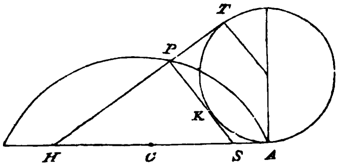
82. If AQC, CPB, BRA be three tangents forming a triangle that circumscribes an ellipse, then

$$AQ.CP.BR = AR.CQ.BP.$$

83. If a segment be cut off from an ellipse by a chord, and any polygon be described about the ellipse, such that the portions of the chord between the sides of the polygon and the curve be equal, the continued products of the alternate segments of the tangents will be equal.

84. If a circle be described intersecting an ellipse in four points, and chords be drawn through the points of intersection, diameters parallel to the chords will be equal.

85. If a circle be described touching SP, HP produced, and the major axis of an ellipse; find the locus of its centre.



Since two tangents to a circle are equal, therefore

$$HT = HA,$$

also $$PT = PK, \quad KS = SA;$$

therefore $$HT + HA = SP + HP + HS$$

$$= 2(AC + SC);$$

therefore $$HA = AC + SC,$$

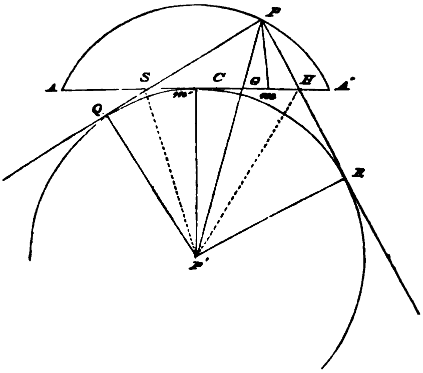
or the circle always touches the axis major at the vertex. The locus of its centre is the tangent at the vertex.

86. The locus of the centre of the circle inscribed touching SP, HP produced, and SH, is an ellipse of which the major axis $= SH$ and minor axis $B'C$, where

$$B'C : BC :: SC : AS.$$

Let P' be the centre of the inscribed circle PR, PM' radii at two of the points of contact, then

$$P'R = P'Q, \quad SC = SQ, \quad M'H = HR;$$



therefore $$SP + PH + SH = PR + PQ$$

$$= 2PR;$$

therefore $$PR = AC + SC,$$

and $$PP'^2 = PR^2 + P'R^2$$

$$= (AC + SC)^2 + P'm'^2,$$

but also $$PP'^2 = mm'^2 + (Pm + P'm')^2;$$

but area of $\triangle PSH$ = area $PQP'R$ − area $QSHRP'$

$= 2\triangle PRP' - 2\triangle SP'H,$

because $HR = Hm'$ and $QS = Sm'$

$= P'R.\{PR - SH\}$

$= P'm\,\{AC - SC\};$

also area of $\triangle PSH = Pm\,.\,SC.$

Therefore $P'm : Pm :: SC : AC - SC;$

therefore $P'm : Pm + P'm :: SC : AC$ .... (A):

also $m'G : mG :: SC : AS$............(1),

$\because AC - SC = AS.$

Therefore $m'm : mG :: AC : AS$

$:: AC.A'S : BC^2, \quad \because AS.A'S = BC^2;$

but $Cm : mG :: SC^2 : BC^2$..............(2),

by (1) and (2), $m'G : CG :: A'S.SC : SC^2;$

using (2) again, therefore,

$m'C : mG :: SC.AC : BC^2;$

and by (1) $m'm : m'C :: A'S : SC$............(B).

Substituting from (A) and (B) in the value of PP'^2,

$$A'S^2 + P'm'^2 = \frac{AC^2}{SC^2}\,.\,P'm'^2 + \frac{A'S^2}{SC^2}\,.\,Cm'^2,$$

or $A'S^2.\,SC^2 = BC^2.\,P'm'^2 + A'S^2.\,Cm'^2;$

therefore $BC^2.P'm^2 = A'S^2.\,(SC^2 - Cm'^2),$

or $P'm^2 :: SC^2 - Cm'^2 :: A'S^2 : BC^2$

$:: BC^2 : AS^2$

$$:: \frac{BC^2.\,SC^2}{AS^2} : SC^2,$$

which proves the proposition.

87. The locus of the centre of the circle inscribed in the triangle PSH is an ellipse whose semiaxes are SC and $\dfrac{BC\,.\,SC}{A'S}$.

88. Circumscribe a circle about a semiellipse cut off by the axis minor. If D be its diameter,

$$D^2 : AC^2 + BC^2 :: AC^2 + BC^2 : AC^2.$$

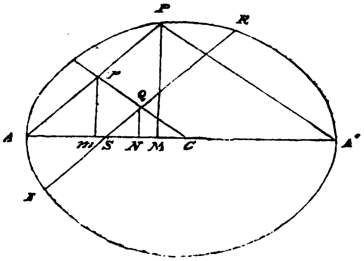
89. The radius of a circle inscribed in a semiellipse touching the minor axis, being R,

$$R : AC :: BC : SC.$$

90. HE and DS are two equal rods moveable about H and S, DE is a string of length equal to HS; as the rods revolve, DE remaining stretched, P the point of their intersection traces out an ellipse whose major axis is equal to HE.

91. If two circles touch each other internally, the locus of the centres of circles touching both is an ellipse whose foci are the centres of the given circles.

92. The locus of the middle points of all focal chords in an ellipse is a similar ellipse.



Let $RQSr$ be any focal chord, draw AP parallel to SQ, and draw CQp bisecting Rr in Q, and AP in p, then $A'P$ is parallel to Cp. Draw PM, pm, QN, ordinates to the major axis; then, by similar triangles Apm, SQN,

$$SN : QN :: Am : pm :: AM : PM;$$

and by similar triangles CQN, $A'PM$,

$$CN : QN :: A'M : PM,$$

therefore $SN.CN : QN^2 :: AM.A'M : PM^2 :: AC^2 : BC^2$,

the property of an ellipse.

Similarly the middle points of chords passing through any point may be shewn to be an ellipse.

93. If a rod slide between a vertical wall and a horizontal plane, any point in it traces out an ellipse.

94. S and H being the foci, P a point in the ellipse, if SH be bisected in L, and AL be drawn from the vertex cutting SP in Q, the locus of Q is an ellipse whose focus is S.

95. If two circles be described with a centre C, and a radius CLP to the outer, cut the inner in L; then if pPM be drawn perpendicular to a diameter and LP parallel to it, P the point of intersection will describe an ellipse.

96. If CDE be a ruler, P a point in DE having a hinge at D and moveable about a fixed point at C; then if E be moved up to C along a straight line CA, the point P traces out an ellipse.

97. If from the centre C of an ellipse a line CQ be inflected on the ordinate Pm, so that CQ may equal Pm, find the locus of the point Q.

The locus is an ellipse whose major axis is the minor axis of the given ellipse and whose major axis is CE, such that

$$CE : AC :: BC : AB.$$

98. If two ellipses have the same foci, the lines joining the points of intersection pass through the intersection of the directrices.

99. If AM be the axis major of the elliptic section of a right cone, KM and AD the diameters of the circular sections at M and A, then $$Am^2 = AK^2 + AD . KM.$$

100. If S and H be the foci of the ellipse in a section, as above, $$SH = AK.$$

101. If CP, CD be any pair of conjugate diameters PM, DN, ordinates at P and D, and a concentric ellipse on the same axis

cut PM, DN in Q and E, CQ and CE will be conjugate diameters.

The lines joining the centre with the points in which PM, DN meet the circle on the axis major, are at right angles to each other,* whence the problem is immediately solved.

102. If the tangent at a point P meet any pair of conjugate diameters in T, t, and CD be conjugate to CP, $PT.Pt = CD^2$.
Emerson's *Conic Sections*, Prop. 53.

If ordinates PM, PN be drawn from P to the conjugate diameters, then

$$PM : CM :: CN : PN,$$

as is proved in most works on the subject: on this property the proof depends.

103. Given two conjugate diameters of an ellipse; find the axes by a geometrical construction.

Let PCp, DCd be the diameters, produce CP to H, making PH such that $CP : CD :: CD : PH$.

Draw TPt a tangent at P, bisect CH in O, draw DR perpendicular to CH intersecting TPt in R; with centre R and radius CR describe a circle cutting TPt in T, t; join CT, Ct cutting the ellipse (if it be traced) in A, B; CA, CB will be the semiaxes, for $CD^2 = CP.PH$

$= PT.Pt$ by property of circle; therefore CA, CB are conjugate diameters, as follows from the last problem, and Tt is a diameter of the circle; therefore the conjugate diameters are at right angles, they are therefore the principal diameters.

If the ellipse be not traced, we may determine AC and BC by dropping PM, PN perpendiculars upon CT and Ct, and taking AC and BC mean proportionals between CT and CM, Ct and CN. Emerson's *Conic Sections*, Prop. 86.

104. For what position of the point P in an ellipse is the angle SPH greatest?

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* Dr. Whewell's *Conic Sections*, *Ellipse*, Prop. 8, Cor. 2.

The point of intersection with the smallest segment of a circle which can be described upon SH so as to meet the curve, will be that required. But the segment which touches the ellipse is less than all which cut it, and B is evidently the point of contact, as it is the only point at which the normal to the ellipse bisects the chord of the segment.

Senate-House Papers, 1848.

105. Of all parallelograms circumscribing an ellipse, those whose sides are parallel to conjugate diameters are the least.

Let PCp, QCq be conjugate diameters, and draw tangents at P and p; then, if the parallelogram be formed by drawing tangents at the extremities of any other diameter than QCq, the diameter QCq being produced will meet those tangents in R, r suppose: then, the parallelogram $= 4.CR.PF$, PF being perpendicular to QCq, which is greater than $4CQ.PF$, whence the truth of the proposition. *Senate-House Papers*, 1848.

106. When is the square of the sum of a pair of conjugate diameters the least?

Let CP, CD be semiconjugate diameters, PE perpendicular to CA produced, then

$$\begin{aligned}(CP + CD)^2 &= CP^2 + CD^2 + 2CP.CD,\\ &= AC^2 + BC^2 + 2CP.CD;\end{aligned}$$

therefore the square of the sum is least when $CP.CD$ is least, and $CP.CD$ is always $> PF.CD$, as is plain from the figure, unless P is at the extremity of one of the principal axes, in which case PF coincides with CP. Hence the semiaxes are the pair of conjugate diameters, the square of whose sum is least. *Senate-House Papers*, 1848.

107. In the ellipse the radius of curvature

$$= \frac{(\text{normal})^3}{\left(\dfrac{\text{latus rectum}}{2}\right)^2},$$

radius of curvature $= \dfrac{CD^2}{PF}$, and $PF.CD = AC.BC$, therefore $CD^2 = \dfrac{AC^2.BC^2}{PF^2}$;

therefore radius of curvature $= \dfrac{AC^2.BC^2}{PF^3}$,

and $PF.PG = BC^2$, G being the foot of the normal,

therefore radius of curvature $= \dfrac{PG^3}{\dfrac{BC^4}{AC^2}}$.

108. The circle of curvature at a point where the conjugate diameters are equal, meets the ellipse again at the extremity of the diameter.

109. The chord of curvature at P perpendicular to the major axis $= \dfrac{2CD^2}{BC^2}PM$, PM being the ordinate at P.

110. Find the points at which the radius of curvature is a mean proportional between the major and minor axes.

111. The chord of curvature passing through the focus, is equal to the focal chord which is drawn parallel to the tangent. This follows from Prob. 55.

112. If S and H be the foci of an ellipse, B the extremity of the axis minor, a circle described through S, H, and B, will cut the minor axis in the centre of curvature at B.

113. If R and R' be the radii of curvature at the extremities of two conjugate diameters in an ellipse,

$$R^{\frac{2}{3}} + R'^{\frac{2}{3}} = \frac{AC^2 + BC^2}{(AC.BC)^{\frac{2}{3}}}.$$

114. If C and C' be the *curvatures* at two points at which the tangents are at right angles,

$$C^{\frac{2}{3}} + C'^{\frac{2}{3}} = \frac{AC^2 + BC^2}{(AC.BC)^{\frac{4}{3}}}.$$

See Prob. 66. Curvature is measured by the reciprocal of the radius of curvature.

Many of the problems which have been given for the ellipse as the standard case, may be proved to hold for any Conic Section; as they depend only on properties possessed by the three curves in common; as, that the distances of the describing point from the focus and the directrix are in a constant ratio;* that the rectangles contained by the segments of parallel chords are so also, and the like.

Several problems may be adapted to the Hyperbola by making the difference of the focal distances constant instead of the sum, or to the Parabola by supposing that the second focus is at an infinite distance, and that in consequence all focal lines drawn to it are parallel. It thus becomes an Ellipse, whose major axis is infinite. The construction for drawing a tangent from a given point (Prob. 29) may thus be applied to the other curves, bearing in mind that in the case of the Parabola, the circle on the major axis becomes the tangent at the vertex, as a circle whose radius is infinite is a straight line.

* Prob. 98 is an instance of this.

THE HYPERBOLA.

A GREAT part of the properties of the Hyperbola are analogous to those of the Ellipse; hence, the following problems will for the most part have reference to the asymptotes, or to the case of the rectangular hyperbola. A rectangular hyperbola has its principal axes equal, whence it follows that the asymptotes are at right angles to each other.

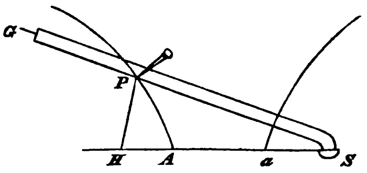
1. To describe a hyperbola practically.

Take two points S and H for foci, at S fasten a point, about which let a rod SPG revolve. At H fasten one end of a string of a length such that the rod exceed it by a quantity equal to the transverse axis of the proposed hyperbola. If then a pencil be placed as at P so as to confine the string against the rod, and be moved so as to keep the string stretched when the rod revolves about S, P will trace out the hyperbola, for

$$GS - (GP + PH) = \text{length of rod} - \text{length of string}$$

$$= Aa, \quad \therefore SP - PH = Aa,$$

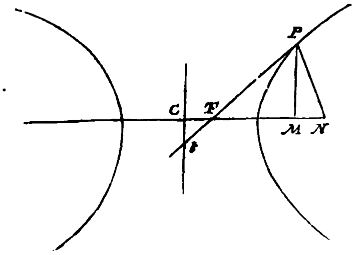
the property of the hyperbola.



2. The locus of the centre of a circle touching two circles externally is a hyperbola.

3. The locus of the centre of a circle touching one circle internally, and one externally, is a hyperbola.

4. If the tangent at P cut off CT, Ct from the axes, then $PT.Pt = CD^2$, CD being the semi-diameter conjugate to CP.



By similar triangles, $Pt : PT :: CM : TM$.

Also $PT^2 = TM.TN$; $\therefore PT : TM :: TN : PT$;

therefore $Pt.PT = CM.TN = CM.(CN - CT)$

$= CM.CN - AC^2$.

But $CM : MN :: CA^2 : CB^2$,

$CM : CN :: CA^2 : CA^2 + CB^2$,

therefore $CM^2 : CN.CM :: CA^2 : SC^2$;

but if a point I be taken so that

$CM : CI :: CA : SC$, then $CI^2 - CA^2 = SP.HP$.

Dr. Whewell's *Conic Sections*, Prop. 2.

Hence $CI^2 = CN.CM$; $\therefore CI^2 - AC^2 = PT.Pt$,

therefore $PT.Pt = SP.HP = CD^2$.

5. If PM be an ordinate, DCE the conjugate axis, then if AD be drawn to the ends of the transverse and conjugate axes, and MQ be parallel to it, then $PM^2 = QD.QE$.

6. If two tangents at P and Q cut off AR, $A'R'$, AL, $A'L'$ from the tangents at the vertices, then $AR.A'R' = AL.A'L'$.

7. If any chord AP through the vertex of an hyperbola be divided in Q, so that $AQ : QP :: AC^2 : BC^2$, and QM be drawn to the foot of the ordinate MP, shew that a straight

line at right angles to QM through the point Q, cuts the transverse axis in the same ratio.

8. If a tangent at any point of an hyperbola cut the tangents at the vertices A and A' in T and T', then $AT.AT' = BC^2$.

9. If a perpendicular drawn from the foot of the normal, intersect the radius vector at a point P in L, then $PL = \frac{\text{latus rectum}}{2}$.

The proof is similar to that of the same proposition in the ellipse, the property $PF.PN = BC^2$ holding in both cases.

10. If a tangent cut the major axis in the point T, and perpendiculars $SY.HZ$ be let fall on it from the foci, then $AT.A'T = YT.ZT$.

11. If on the portion of any tangent intercepted between the tangents at the vertices a circle be described, it will pass through the foci.

12. If PM be an ordinate drawn from a point P in the hyperbola, MQ a tangent to the circle on the major axis, and PN parallel to QC; then $MN = BC$.

13. If a hyperbola and an ellipse have the same foci, they cut one another at right angles.

A tangent to the one is easily shewn by the property of the angles between the tangent and focal distances to be a normal to the other.

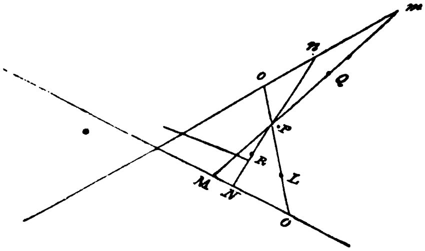
14. If a circle be inscribed in the triangle SPH, the locus of its centre is the tangent at the vertex.

15. If a circle be described touching SP, HP produced and the major axis, the locus of its centre will be a hyperbola. See *Ellipse*, Problems 85, 86.

16. If in two concentric hyperbolas two points be taken, whose abscissæ are as their major axes, the locus of the middle

point of the line joining them is a hyperbola, concentric, and similarly situated, whose axes are arithmetic means between those of the given hyperbolas.

17. Given the asymptotes and a point in the curve, to find any number of points.



Let P be the given point, through P draw any number of lines,

$$mQPm, \quad nPRN, \quad oPLO, \text{ \&c.};$$

and make $Qm = PM$, $RN = Pn$, $LO = Po$, &c.,

then Q, R, L, &c. are points in the curve, and through each of these points lines may be drawn as through P, and more points determined.

18. If from any point P in the hyperbola, $RPQS$ be drawn meeting the hyperbola in P and Q, and the asymptotes in R and S, then PK and QL being drawn parallel to one asymptote to meet the other, $LS = PK$.

19. If from the point P in a hyperbola, PK be drawn parallel to the transverse axis cutting the asymptotes in I and K, then rectangle $KP.PI = CA^2$.

20. If from the points P and Q in a hyperbola there be drawn PR, QS parallel to each other to meet one asymptote, and PO, QI also parallel to each other to meet the other asymptote, then the rectangle $PO.PH$ equals rectangle $QI.QS$.

21. If any two tangents be drawn to a hyperbola, and their intersections with the asymptotes be joined, the joining lines will be parallel.

22. If a chord PQ to a hyperbola intersect the asymptotes in R and S, and from the point R a tangent be drawn to the hyperbola in the point E, then PM, QN, EL being drawn to one asymptote parallel to the other, $2EL$ equals $PM + QN$.

23. If from a point P in the hyperbola, PN be drawn parallel to an asymptote to meet the directrix in N, then $PN = SP$.

24. If the line PN drawn as above, meet the tangent at the vertex in R, then

$$SP = PR + \frac{L}{2}.$$

25. The part of the tangent to a hyperbola intercepted between the asymptotes is equal to the diameter at the point of contact.

26. If from any point P in a hyperbola, PD be drawn parallel to an asymptote, to meet the conjugate hyperbola in D, then CP and CD are conjugate diameters.

27. If a circle be described from the focus of a hyperbola, with a radius equal to the semi-axis minor, it will touch the asymptotes at the points where they are cut by the directrix.

28. If A and S be the vertex and focus of a hyperbola, and the tangent at the vertex and the directrix meet the asymptote in G and R respectively, then SG is parallel to AR.

29. If an ellipse and hyperbola have the same foci, and tangents be drawn to the one to intersect at right angles those drawn to the other, the locus of the points of intersection is a circle whose $(\text{radius})^2$ is equal to $A'C^2 + BC^2$, where $A'C$

is the major axis of the hyperbola, BC the minor axis of the ellipse.

The method of the analogous case in the ellipse (Prob. 64) will be useful.

30. Given an asymptote, the focus, and a point; describe the hyperbola. The property of Prob. 23 affords an easy construction.

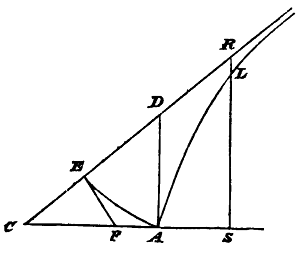
31. Given an asymptote, the directrix, and a point; construct the hyperbola.

32. Given two conjugate diameters of a hyperbola; find the axes and construct the curve.

Bisecting the angles between the diameters, we find the directions of the asymptotes; and bisecting the angle between them, we determine the directions of the axes: then, drawing tangents at the extremities of the diameters, we find AC, BC by the properties $CT.Cm = AC^2$, $Ct, CN = BC^2$.

33. Describe a circle touching the hyperbola and the asymptotes, and shew that its radius equals the part of the ordinate intercepted between the extremity of the latus rectum and the hyperbola.

Draw AD a tangent at the vertex A meeting the asymptote in D, with centre D and radius DA describe a circle cutting the asymptotes in E, let the tangent to this circle at E cut the axis in F; then, since $FA = FE$ and FE is perpendicular to the asymptote, F is the centre and FE the radius of the circle required; also



$$DE = DA = BC,$$

and $$DC = SC, \therefore CE = SC - BC,$$

and $$EF : SC - BC :: BC : AC;$$

$$\therefore EF.AC = SC\,BC - BC^2, \text{ but } SR : SC :: BC : AC;$$

$$\text{therefore } SR.AC = SC.BC,$$

$$\text{and } SL.AC = BC^2;$$

therefore $(SR - SL)\,AC = SC.BC - BC^2 = ET.AC$,

therefore $EF = SR - RL = RS.$ Q. E. D.

The rectangular hyperbola is often called the *equilateral* hyperbola.

34. In a rectangular hyperbola all conjugate diameters are equal to one another.

35. In a rectangular hyperbola no pair of tangents can be drawn at right angles to each other.

36. In a rectangular hyperbola the eccentricity is the ratio of the diagonal of a square to the side.

37. In a rectangular hyperbola, if a perpendicular from the focus meet the asymptote in B, then $CB = CA$.

38. If S and H be the foci of a rectangular hyperbola, and a circle be described on SH, the tangents at the vertices will intersect the asymptotes in the circumference.

39. In a rectangular hyperbola the distance of P from the centre is a geometric mean between its distances from the foci.

40. In a rectangular hyperbola, centre C, CP equals the length of the normal at P.

41. If two concentric rectangular hyperbolas be described, the axes of one being the asymptotes of the other, they will intersect at right angles.

42. If a rectangular hyperbola be cut at a point P by a concentric circle, and PT, PT' be tangents to the hyperbola and circle cutting the asymptotes in T and T', and PM be perpendiculars to the asymptote, then $MT.MT' = PM^2$.

43. If from a point Q in the conjugate axis of a rectangular hyperbola QA be drawn to the vertex, and QP parallel to the transverse axis to meet the curve, $QP = AQ$.

Hence we see that if one extremity of an isosceles triangle be fixed, and the vertex lie in a straight line, so that the other side may be always perpendicular to that line, the other extremity of the base traces out a rectangular hyperbola.

44. If P be the middle point of a line AB which is so drawn as to cut off a constant area from the corner of a square, its locus is a rectangular hyperbola.

45. Given a point Q and a straight line AB, if a line QCP be drawn cutting AB in C, and P be taken in it, so that PD being a perpendicular upon AB, CD may be of constant magnitude; find the locus of P.

Draw OB' perpendicular to AB, making $AO = CD$, then the locus is a rectangular hyperbola whose centre is O.

46. If S and H be the foci of a rectangular hyperbola and a circle be described upon SH, then the quadrantal chord of this circle shall be a tangent to that described upon the major axis.

47. If G be the foot of the normal at a point P in the rectangular hyperbola R, S, and T, the points where the tangent at P cuts the asymptotes and the transverse axis, then $CT.TG = ST.TR$.

The properties of (40) and of (34) are of use here.

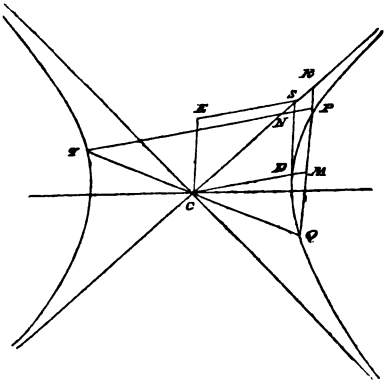
48. If from the point G where the normal at P meets the axis, GR, GS be drawn to the points where the tangent at P intersects the asymptotes, the angle RGS is a right angle.

49. GR, GS being as above,

$$RC.SG + RG.CS = RS.CG.$$

50. If from any point in a rectangular hyperbola lines be drawn to the extremities of a diameter, these lines will make equal angles with the asymptotes.

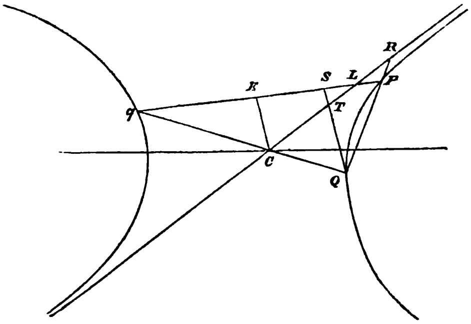
Let Qq be the given diameter, P the given point. Draw CD and CE bisecting PQ and Pq, they will be conjugate diameters, for $Cq = CQ$; therefore CM, CE bisect PQ, Pq, and tangents at D and E will intersect at some point S on CR, and as



$$RP : PN :: RM : CM :: SD : CD$$
$$:: CE : CD;$$

but $CE = CD$, $\therefore RP = PN$, $\therefore \angle NRP = \angle RNP$. Q. E. D.

51. If lines be drawn from any point of a rectangular hyperbola to the extremities of a given diameter, the difference between the angles which they make with the diameter will be equal to the angle which it makes with its conjugate.



For greater clearness in the figure, only one asymptote is drawn.

Draw CE conjugate to the given diameter QCq, and QTS a tangent at Q, then the angle

$$SLT = \angle RLP$$
$$= \angle LRP \text{ by Prob. 50.}$$

Also $QT = CQ$, $\therefore \angle CTQ = \angle TCQ$, $\therefore \angle STL = \angle RCQ$,

therefore the triangles LST, RCQ are equiangular;

therefore $\angle LST = \angle SQP$:

hence in the triangles PSQ, PqQ, the angle $PSQ = \angle PQq$,

and also $\angle PqQ = \angle SQP$,

therefore $\angle SQC = \angle PQq - \angle PqQ = \angle qCE$. Q. E. D.

52. Upon a given base, construct a triangle whose vertical angle and the difference of the angles at the base are given.

This may be done by the intersection of two loci, by means of the last problem.

53. If AA' be any diameter of a circle, PQ any ordinate to it, then the locus of the intersections of AP, $A'Q$ is a rectangular hyperbola.

54. If a series of isosceles right-angled triangles have the same vertex and one extremity of their bases in a given straight line, the locus of the other extremity will be a rectangular hyperbola.

55. If between a rectangular hyperbola and its asymptotes, any number of concentric elliptic quadrants be inscribed, the rectangle contained by their axes will be invariable.

56. Given an asymptote and a tangent at a given point, construct a rectangular hyperbola.

57. In a rectangular hyperbola, focal chords parallel to conjugate diameters are equal.

58. If a series of lines have their extremities in two straight lines at right angles to one another, and all pass through

a given point, the locus of their middle points is a rectangular hyperbola.

59. If PR be any chord, Q the point of contact of the tangent parallel to it, then if PD, QE, RH be drawn parallel to one asymptote to meet the other, CD, $CH = CE^2$.

60. If through P and Q any points in a hyperbola PD, QE be drawn parallel to one of the asymptotes, the hyperbolic sector PCQ is equal to the hyperbolic trapezium $PDEQ$.
Wallace's *Conic Sections*, Prop. 31.

61. If CD, CE, CH be taken along one asymptote in continued proportion, and DP, EQ, HR be drawn parallel to the other asymptote to meet the curve, the hyperbolic areas $PDEQ$, $QEHR$ are equal.
Wallace's *Conic Sections*, Prop. 32.

Produce PR to meet the asymptotes, and make use of the property of the equality of the intercepts as well as of that of (59).

62. In a hyperbolic section of a right cone, A and A' being the vertices of the sections AE, and $A'F$ the diameters of the sections perpendicular to the axis; $AE . A'F = AA'^2$.

63. Shew that the shadow of the circular horizontal rim of a lamp cast upon a vertical wall is a hyperbola, and having given the radius of the rim, its height above the flame, and its distance from the wall, construct the hyperbola, and determine its major and minor axes.

The tip of the flame is the vertex of a cone of which the rim forms a circular section, the wall is a section parallel to a plane through the axis of the cone; if h be the height, r the radius, d the distance, then the major axis $= \frac{hd}{r}$, and the minor axis $= d$.

64. Each of a series of parabolas is described through two given points, and has its axis parallel to a given fixed line. Prove that the locus of the foci is a hyperbola.

Let S and H be the given points, SK, HK' perpendiculars from them upon the directrix of any one of the parabolas whose focus is P; then, $SP - HP = SK - HK'$, a constant quantity for all the series of parabolas. Hence the locus of P is a hyperbola whose foci are S and H, and whose major axis equals $SK - HK'$.

65. If a parabola be described on an asymptote of a rectangular hyperbola for axis, with its vertex at the centre, the ordinate of the parabola at the point of intersection produced will pass through the focus of the hyperbola.

66. The curve which trisects the arcs of all segments of a circle described upon a given base, is a hyperbola whose eccentricity is 2.

Let SH be the given base, describe upon it any number of segments of circles, bisect SH in C, and draw CY of indefinite length perpendicular to SH. With S as focus, and CY for directrix, describe a hyperbola of eccentricity 2, cutting any one of the segments in P, draw PN perpendicular to CY.

Then $Pn = \frac{1}{2}PS$, therefore if Pn produced meet the segment again in Q, the chords SP, PQ, QH, will be equal; hence the segment is trisected in Q and P, which points lie in a hyperbola, and so with all the segments: hence the truth of the proposition. *Senate-House Papers*, 1848.

67. With two conjugate diameters of an ellipse as asymptotes, a pair of conjugate hyperbolas is constructed: prove that if one hyperbola touch the ellipse, the other will do so likewise; prove also that the diameters drawn through the points of contact are conjugate to each other.

Let P be a point of contact, draw through it a common tangent to the curves, terminated by the conjugate diameters produced in T, t. Then, by the property of Prob. 102 in the *Ellipse*, if CD be the conjugate to CP, $CD^2 = PT.Pt$; but P being a point in the hyperbola, the tangent at which meets the asymptotes in T, t, $CD^2 = PT \cdot Pt$, CD being the conjugate to CP, and D is a point in the conjugate hyperbola as well as in the ellipse, the tangent at which in both cases is parallel to CP: hence the curves touch at the extremities of conjugate diameters. Q. E. D.

Senate-House Papers, 1848.

68. In the rectangular hyperbola, the radius of curvature at a point varies as CP^3.

69. In the rectangular hyperbola, if CP be produced to Q, so that $PQ = CP$, and QO be drawn perpendicular to CQ to intersect the normal in O, O is the centre of curvature at P.

70. If PK, PK' be the chords of curvature in a rectangular hyperbola parallel to the asymptotes,

then $$PK \cdot PK' = CP^4 \cdot \left\{\frac{1}{AC^2} + \frac{1}{BC^2}\right\}.$$